C207 problem set 2 Solutions

1 Limb Darkening and Exoplanet Transits

(this solution from Jennifer Barnes and Michelle Galloway).

We study limb darkening by solving the radiation transport equation in plane parallel coordinates, assuming gray opacity:

$$\mu \frac{\partial I(\tau_z, \mu)}{\partial \tau_z} = I(\tau_z, \mu) - S(\tau_z, \mu). \tag{1}$$

We take as an ansatz that the angular dependence takes the form

$$I(\tau_z, \mu) = I_0(\tau_z) + I_1(\tau_z)\mu, \tag{2}$$

where

$$I_0 > I_1 > 0 \quad \forall \quad \tau_z.$$

b Mean intensity, flux, energy density, and radiation pressure

b.1 Mean intensity

In our coordinate system,

$$\mathrm{d}\Omega = \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi = -\,\mathrm{d}\mu\,\mathrm{d}\phi,$$

so

$$J = \frac{1}{4\pi}\oint I\,\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}(I_0 + I_1\mu)\,\mathrm{d}\mu\,\mathrm{d}\phi.$$

Evaluating the expression gives

$$J = I_0(\tau_z).$$

b.2 Flux

$$\begin{aligned} F &= \oint I\,\cos(\theta)\,\mathrm{d}\Omega \\ &= \int_0^{2\pi}\int_{-1}^{1}(I_0\mu + I_1\mu^2)\,\mathrm{d}\mu\,\mathrm{d}\phi. \\ &= \frac{4\pi I_1(\tau_z)}{3} \end{aligned}$$

b.3 Energy density

$$u(\tau_z) = \frac{4\pi}{c} J(\tau_z) = \frac{4\pi I_0(\tau_z)}{c}$$

b.4 Radiation pressure

$$p = \frac{1}{c}\int_0^{2\pi}\int_{-1}^{1} (I_0 + \mu I_1)\mu^2 \,\mathrm{d}\mu\,\mathrm{d}\phi = \frac{4\pi I_0}{3c}.$$

The ratio of radiation pressure to energy density is 1:3, the same as for isotropic radiation.

c Integrating the zeroth moment of the radiation transport equation

The zeroth moment is simply Eq. (1) with $I(\tau_z)$ taking the form given in Eq. (2). This is integrated over all solid angles:

$$\begin{aligned} \mu\frac{\partial I}{\partial \tau} &= I - S \\ \oint \mu\frac{\partial}{\partial \tau_z}(I_0 + \mu I_1)\,\mathrm{d}\Omega &= \oint I_0 + \mu I_1 - S\,\mathrm{d}\Omega. \end{aligned}$$

We carry out the integration, making the assumption that $S \neq S(\mu)$, and find that

$$\frac{\partial}{\partial \tau_z}\frac{2}{3} I_1 = 2I_0 - 2S. \tag{3}$$

But

$$\frac{\partial}{\partial \tau_z}\left(\frac{2}{3} I_1\right) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_z} F = 0,$$

since $F = \sigma T_{eff} \neq F(\tau_z)$. From earlier, we have the result that $J = I_0(\tau_z)$, so we can rewrite Eq. (3):

$$\begin{aligned} 0 &= 2J - 2S \\ \Rightarrow J &= S. \end{aligned}$$

d Integrating the first moment of the radiation transport equation

Multiplying the zeroth moment by μ and integrating over solid angles gives:

$$\int_0^{2\pi}\int_{-1}^{1} \mu^2 \frac{\partial}{\partial \tau_z}(I_0 + \mu I_1)\,\mathrm{d}\mu\,\mathrm{d}\phi = \int_0^{2\pi}\int_{-1}^{1} \mu I_0 + \mu^2 I_1 - \mu S\,\mathrm{d}\mu\,\mathrm{d}\phi,$$

which simplifies nicely:

$$\frac{\partial I_0}{\partial \tau_z} = I_1.$$

From a), we know that $I_1 = \frac{3}{4\pi}F = \frac{3}{4\pi}\sigma T_{eff}^4$. Plugging this in yields:

$$I_0 = \int I_1\,\mathrm{d}\tau_z = \frac{3}{4\pi}\sigma T_{eff}^4 \tau_z + C,$$

where C is an integration constant. We can now write out an expression for the specific intensity up to a constant of integration:

$$I(\tau_z, \mu) = I_0 + \mu I_1 = \frac{3}{4\pi}\sigma T_{eff}^4 \left(\tau_z + \mu + \tilde{C}\right).$$

e Determining the integration constant

Apply the approximation $F_{inward}|_{\tau_z=0} = 0$. This gives:

$$\begin{aligned}
0 = \int_{inward} I\cos(\theta)\,\mathrm{d}\Omega &= \frac{3}{4\pi}\sigma T_{eff}^4 \int_0^{2\pi}\int_{-1}^{0} \mu^2 + \tilde{C}\mu\,\mathrm{d}\mu\,\mathrm{d}\phi \\
&= \frac{3}{2}\sigma T_{eff}^4 \left(\frac{\mu^3}{3}\bigg|_{-1}^{0} + \tilde{C}\cdot\frac{\mu^2}{2}\bigg|_{-1}^{0} \right) \\
&= \frac{3}{2}\sigma T_{eff}^4 \left(\frac{1}{2} - \frac{\tilde{C}}{2}\right) \\
\Rightarrow \tilde{C} &= \frac{2}{3}.
\end{aligned}$$

We now have a full expression for the specific intensity:

$$I(\tau_z, \mu) = \frac{3}{4\pi}\sigma T_{eff}^4 \left(\tau_z + \mu + \frac{2}{3}\right).$$

Plotted in Figure 1 is the emergent intensity:

$$\frac{I(\tau = 0, \mu)}{I(\tau_z = 0, \mu = 1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right).$$

The atmosphere does appear to be radiative.

f Expression for the transit light curve

f.1 Approximations

1. We assume $R_p << R_s$
2. Because the orbital radius is so much greater than the star, we approximate the planet's trajectory to be linear (i.e. $\frac{d}{dt}\sin(\theta)$, and not $\frac{d}{dt}\theta$ is constant, where θ measures the angle between the observer's line of sight and the line from the stellar center to the planet.) See Figure 1.
3. Because the observer is so far from the star/planet system, we take $\mu_{observer} = \cos(\theta)$. In other words, we take the angular variation over the surface of the star, relative to the observer, to be negligible. (See Figure 2.)

f.2 Flux due to the star

We calculate the total flux from the star without interference from the planet. Keeping in mind

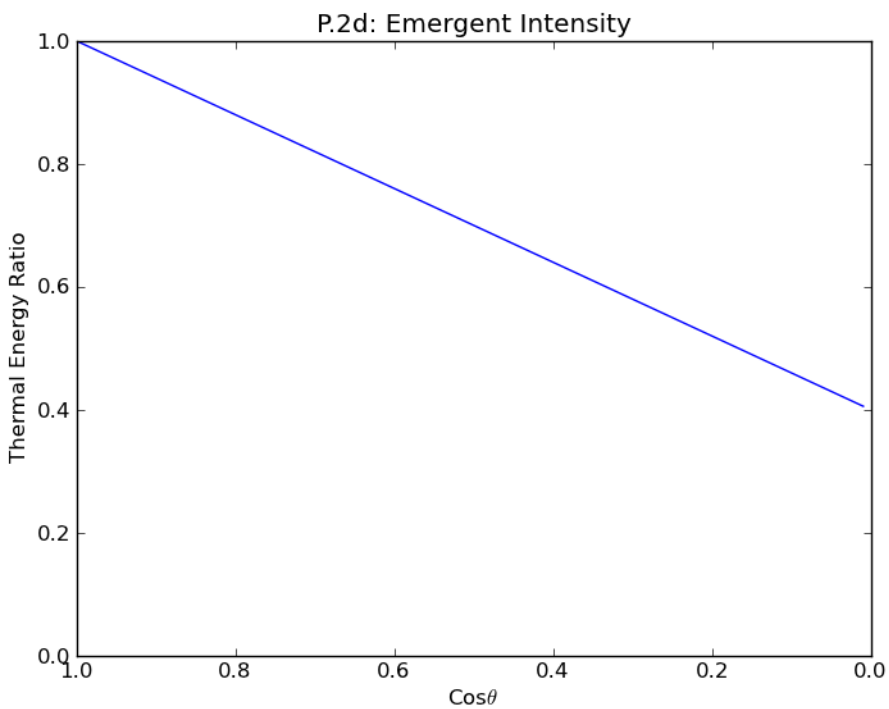


Figure 1: Limb darkening profile

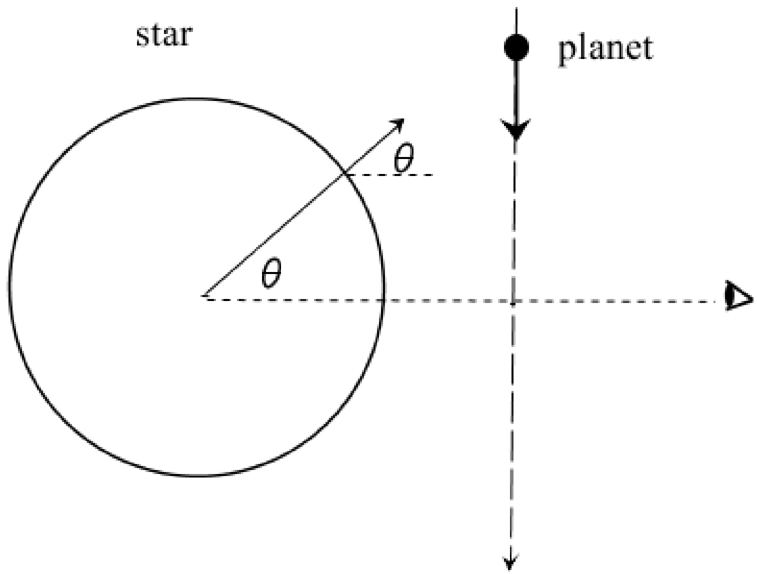


Figure 2: Diagram for the the transit light curve

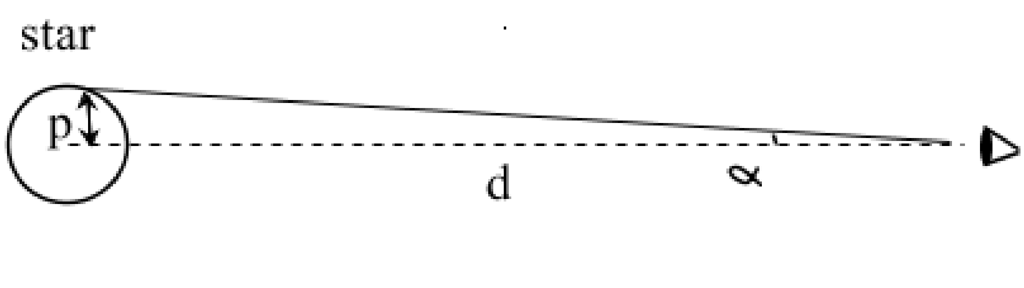


Figure 3: Diagram for calculating the flux of the star

Approximation 3, we write:

$$
\begin{aligned}
F_0 &= \oint I(\mu)\cos(\alpha)\,\mathrm{d}\Omega \\
&= 2\pi \int I(\mu)\cos(\alpha)\,\mathrm{d}(\cos(\alpha)).
\end{aligned}
$$

Some basic geometry allows us to rewrite the integral:

$$
\begin{aligned}
\cos(\alpha) = \frac{d}{\sqrt{d^2+p^2}} = \frac{1}{\sqrt{1+\frac{p^2}{d^2}}} &\simeq 1 - \frac{p^2}{2d^2} \\
\Rightarrow \cos(\alpha) &\simeq 1 \\
\Rightarrow \mathrm{d}(\cos(\alpha)) &\simeq \frac{p\,\mathrm{d}p}{2d^2}.
\end{aligned}
$$

We also rewrite our expression for intensity:

$$
\begin{aligned}
\mu(p) = \cos(\theta(p)+\alpha(p)) \simeq \cos(\theta(p)) &= \sqrt{1-\left(\frac{p}{R_\star}\right)^2} \\
\Rightarrow I(\mu) \to I(p, \tau_z = 0) &= A\left(\frac{2}{3} + \sqrt{1-\left(\frac{p}{R_\star}\right)^2}\right),
\end{aligned}
$$

where A is a known constant (see Part d.) We are finally ready to calculate flux!

$$
\begin{aligned}
F_0 &= \frac{2\pi A}{d^2}\int_0^{R_\star}\left(\frac{2}{3} + \sqrt{1-\left(\frac{p}{R_\star}\right)^2}\right)p\,\mathrm{d}p \\
&= \frac{4\pi A}{3}\frac{R_\star^2}{d^2}
\end{aligned}
$$

f.3 The flux blocked by the planet

It's an analogous setup, but we ignore the variation in I over the planet's surface.

$$
\begin{aligned}
F_p &= 2\pi I(\mu)\int \cos(\alpha)(-\,\mathrm{d}\cos(\alpha)) = 2\pi I(p)\int_0^{R_p}\frac{p\,\mathrm{d}p}{d^2} \\
&= \frac{\pi A}{d^2}\left(\sqrt{1-\frac{p^2}{R_\star}} + \frac{2}{3}\right)R_p^2.
\end{aligned}
$$

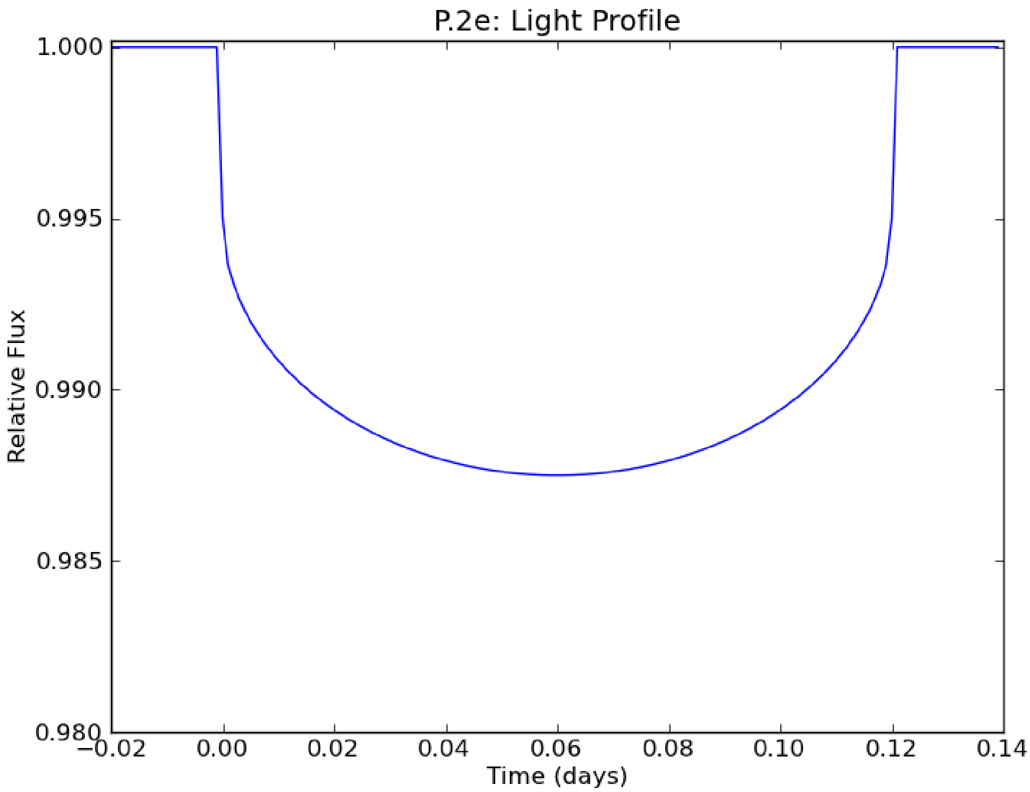
f.4 Relative flux, as a function of time

The relative flux is given by:

$$
F_{rel}(p) = \frac{F_p - F_0}{F_0} = 1 - \frac{3}{4}\left(\frac{R_p}{R_\star}\right)^2\left(\sqrt{1-\frac{p}{R_\star}^2} + \frac{2}{3}\right).
$$

To convert to a function of time, we express impact parameter p as a function of t and the transit time T:

$$
\begin{aligned}
p(t) &= R_\star \,\mathrm{abs}\left(1 - \frac{2t}{T}\right) \\
\Rightarrow F_{rel}(t) &= 1 - \frac{3}{4}\left(\frac{R_p}{R_\star}\right)^2\left(\sqrt{\frac{4t}{T}\left(1-\frac{t}{T}\right)} + \frac{2}{3}\right).
\end{aligned}
$$



To plot this, we take $T = .12$ days, and $\frac{R_p}{R_\star} \simeq \frac{R_{Jupiter}}{R_\odot} \simeq .1$. The plot looks reasonable, given our simplifications.

g The sun's red edge

We make the additional assumption that the atmosphere everywhere is in local thermal equilibrium, so $S(\tau) = B(\tau)$. The radiation transport equation for LTE is:

$$\mu \frac{\partial I}{\partial \tau_z} = I - B(T).$$

Plugging in our expressions for I and $B(T)$ gives:

$$T(\tau_z) = T_{eff} \left(\frac{3}{4}\tau_z + \frac{1}{2} \right)^{\frac{1}{4}}.$$

We assume that we "see" the temperature at a line of sight optical depth $\tau = \frac{2}{3}$. At the center of the sun, $\mu = 1$. The line of sight is radially inward, so $\tau_z = \tau = \frac{2}{3}$.

$$\rightarrow T\left(\frac{2}{3}\right) = T_{eff} \times 1 = 5800\,\mathrm{K}.$$

At the edge of the sun, $\mu = 0$. We take $\tau_z \simeq 0$, so

$$T(0) = T_{eff} \left(\frac{1}{2}\right)^{\frac{1}{4}} \simeq 4880\,\mathrm{K},$$

which is almost 1000 K cooler, thus explaining why the the edge of the sun appears redder.

h Dusty Tori

h.1 Optically Thin limit

(this solution by Sedona Price and Isaac Shivvers)

a) The luminosity of a blackbody sphere is simply

$$L_{\mathrm{bh}} = 4\pi R^2 \sigma_{\mathrm{sb}} T_s^4 \tag{4}$$

Assuming the luminosity is equal to the eddington luminosity of the given mass, and solving for T_s gives,

$$T_s = \left(\frac{c^5}{400 \sigma_{SB} G M_{BH} \kappa_{es}} \right)^{1/4}$$

$$\boxed{T_s = 2.1 \times 10^5 \mathrm{K}}$$

Using Wien's law $\lambda_{pk} T = 0.3$ cm K, we find the accreting black hole radiates most at about

$$\boxed{\lambda_{pk} \approx 142 \text{ Å}} \qquad \text{(EUV)}$$

(b) Now we want to write down the mean intensity $J(r)$ for all $r \geq R_{in}$ and use this to solve for the temperature profile $T(r)$, assuming the envelope is in radiative equilibrium.

$$J = \frac{1}{4\pi} \oint I_0 \, d\mu \, d\phi$$

In this case, I is only non-zero for solid angles looking at the BH radiative surface.

As in lecture on 19 Jan, for the Lambert sphere case (ie our isotropically emitting sphere), we have

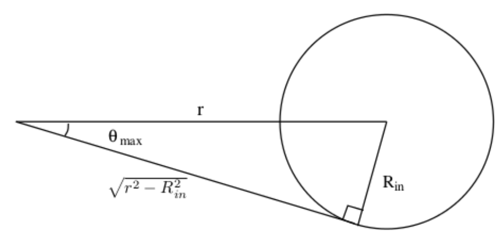
$$I = \begin{cases} I_0 & \theta < \theta_{max} \\ 0 & \theta > \theta_{max} \end{cases}$$

So

$$\begin{aligned} J &= \frac{1}{4\pi} \int_0^{\theta_{max}} \int_0^{2\pi} I_0 \, d\phi \, \sin\theta \, d\theta \\ &= \frac{2\pi}{4\pi} I_0 (-\cos\theta) \Big|_0^{\theta_{max}} \\ &= \frac{1}{2} I_0 \left(1 - \cos\theta_{max}\right) \end{aligned}$$

But

$$\begin{aligned} \cos\theta_{max} &= \frac{\sqrt{r^2 - R_{in}^2}}{r} \\ &= \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2} \end{aligned}$$



So

$$J(r) = \frac{1}{2} I_0 \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2} \right)$$

where $I_0 = B(T_s) = \sigma_{SB} T_s^4 / \pi$, so we get

$$\boxed{J(r) = \frac{\sigma_{SB} T_s^4}{2\pi} \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2} \right)}$$

Now if our dusty envelope is in radiative equilibrium, then $\dot{H}_\gamma = \dot{C}_\gamma$. Here, we're assuming the extinction is grey (and purely absorptive). Grey α implies that $T_{equil} = \left[\frac{\pi}{\sigma_{SB}} J\right]^{1/4}$

So $T(r) = \left[\frac{\pi}{\sigma_{SB}} J(r)\right]^{1/4} \qquad \Rightarrow$

$$T(r) = \left[\frac{\cancel{\pi}}{\cancel{\sigma_{SB}}} \frac{\cancel{\sigma_{SB}} T_s^4}{2\,\cancel{\pi}} \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)\right]^{1/4}$$

$$\boxed{T(r) = T_s \left[\frac{1}{2} \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)\right]^{1/4}}$$

Characteristic temperature for the envelope:

$$\boxed{T(r \sim R_{out}/2) = 210 \text{ K}},$$

and using Wien's law,

$$\boxed{\lambda_{pk} = 14\ \mu\text{m}} \qquad \text{(IR)}$$

(c) Check if the stationary approximation is okay.

For our optically thin dusty envelope,

$$t_{esc} \approx \frac{R}{\lambda_{mfp}} \frac{\lambda_{mfp}}{c} \approx \frac{R_{out}}{c}, \text{ since } R_{out} - R_{in} \approx R_{out} \text{ (ie } t_{esc} \approx \text{light-crossing time)}$$

$$\text{So} \quad \boxed{t_{esc} \approx \frac{R_{out}}{c} = 10^9 \text{ s} \approx 31.7 \text{ yr}}$$

Compare t_{esc} to the dynamical timescale: $t_{dyn} = t_{ff}$. The free fall time (assuming $M_{envelope} \ll M_{BH}$) is found using

$$\ddot{r} = \frac{-GM_{BH}}{r^2}$$

But we know that $\ddot{r} \approx r/{t_{ff}}^2$ (to OOM), so

$$\frac{r}{{t_{ff}}^2} \approx \left|\frac{GM_{BH}}{r^2}\right|$$

$$t_{ff} \approx \sqrt{\frac{{R_{out}}^3}{GM_{BH}}}$$

$$\text{So} \quad \boxed{t_{dyn} = t_{ff} \approx 4.5 \times 10^{12} \text{ s} \approx 1.4 \times 10^5 \text{ yr}}$$

So we find that $t_{dyn} \gg t_{esc}$.

Now consider t_{eq}, the time for the envelope to come into radiative equilibrium. As an estimate, use $t_{eq} \approx t_{cooling}$ or $t_{heating}$ at $\sim R_{out}/2$.

For the thermal cooling time, if we assume that the cooling process is through electron scattering, then $\sigma_{cooling} = \sigma_T$, and if we assume that we have an isotropic, grey, ideal gas envelope, then

$$t_{cooling} \approx \frac{3}{8} \frac{k}{\sigma \sigma_{SB} T^3}$$

$$\text{So} \quad t_{cooling}(R_{out}/2) \approx \frac{3}{8} \frac{k}{\sigma_T \sigma_{SB} T(R_{out}/2)^3}$$

$$\boxed{t_{cooling}(R_{out}/2) \approx 1.47 \times 10^5 \text{ s} \approx 5 \times 10^{-3} \text{ yr}}$$

and we find that $t_{cooling} \lll t_{dyn}$. Thus it does seem safe to assume the envelope structure is fixed over the timescale on which photons are escaping: thermal perturbations are very quickly washed out compared to any collapse, and the photons escape long before the envelope can appreciably collapse.

Optically thick limit: $\tau_0 \gg 1$

For the optically thick case, we need to use the diffusion approximation. We will still assume the envelope opacity is grey and purely absorptive.

The diffusion equation in spherical coordinates is

$$L(r) = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial}{\partial r} u(r)$$

Assuming we have radiative equilibrium, we know that $\frac{\partial}{\partial r}(r^2 F) \equiv$ const, or rather that $L = 4\pi r^2 F \equiv$ const with radius. So we can say that $L(r) = L_{BH} = L_{Edd,es}$ everywhere.

(d) To find the escape time t_{esc} for the optically thick case, we know that photon dispersion $\propto N^2$, where $N \approx R/\lambda_{mfp}$. So

$$t_{esc} = (\text{dispersion}) \frac{\lambda_{mfp}}{c} = \frac{R^2}{{\lambda_{mfp}}^2} \frac{\lambda_{mfp}}{c} = \frac{R^2}{\lambda_{mfp} c} = \frac{R^2 \alpha}{c}$$

where $\alpha = n\sigma = \rho\kappa$. Since $\tau_0 = \rho_0 \kappa R_{out} = \alpha R_{out}$, we can write $\rho_0 = \tau_0/(\kappa R_{out})$, so $\alpha = \tau_0/R_{out}$. So t_{esc} becomes

$$t_{esc} = \frac{{R_{out}}^2}{c} \frac{\tau_0}{R_{out}}$$

$$\boxed{t_{esc} = \frac{R_{out}}{c} \tau_0}$$

At what value of τ_0 does our stationary approximation become questionable? For the stationary approximation to be valid, $t_{esc} \leq t_{dyn}$, and we know $t_{dyn} \approx 4.5 \times 10^{12}$ s $= 1.4 \times 10^5$ yr. We also know that $R_{out}/c \approx 10^9$ s, so $t_{esc} \approx (10^9 \text{ s})\, \tau_0$. So the stationary approximation breaks down if

$$\boxed{\tau_0 \geq 4.5 \times 10^3}$$

(e) Now we solve the diffusion equation for our optically thick case to determine $T(r)$.

$$L(r) = L_{BH} = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial}{\partial r} u(r)$$

Now $\kappa\rho = \alpha = \tau_0/R_{out} \rightarrow \rho = \tau_0/(R_{out}\kappa)$. Given the setup of our problem, we're assuming that ρ is constant. So

$$L_{BH} = -4\pi r^2 \frac{c R_{out}}{3\tau_0} \frac{\partial}{\partial r} u(r)$$

$$\frac{\partial}{\partial r} u(r) = -\frac{3\tau_0}{c R_{out}} \frac{L_{BH}}{4\pi r^2}$$

Integrate with respect to r:

$$u(r) = \frac{3\tau_0}{cR_{out}}\frac{L_{BH}}{4\pi r} + C$$

(where we will specify C later with our radiative zero boundary condition).

Now we know that $u(R) = 4\pi/cJ(r)$, so $J(r) = c/(4\pi)u(r)$, and since for grey absorptivity, we have

$$\begin{aligned} T_{equil}(r) &= \left[\frac{pi}{\sigma_{SB}}J(r)\right]^{1/4} \\ &= \left[\frac{\cancel{\pi}}{\sigma_{SB}}\frac{c}{4\cancel{\pi}}u(r)\right]^{1/4} \end{aligned}$$

$$T_{equil}(r) = \left[\frac{c}{4\sigma_{SB}}\left(\frac{3\tau_0}{cR_{out}}\frac{L_{BH}}{4\pi r} + C\right)\right]^{1/4}$$

Boundary condition: $T_{equil}(R_{out}) = 0$, so $C = -3\tau_0 L_{BH}/(4\pi c {R_{out}}^2)$. Now we get

$$T_{equil}(r) = \left[\frac{\cancel{c}}{4\sigma_{SB}}\left(\frac{3\tau_0}{\cancel{c}R_{out}}\frac{L_{BH}}{4\pi r} - \frac{3\tau_0}{\cancel{c}R_{out}}\frac{L_{BH}}{4\pi R_{out}}\right)\right]^{1/4}$$

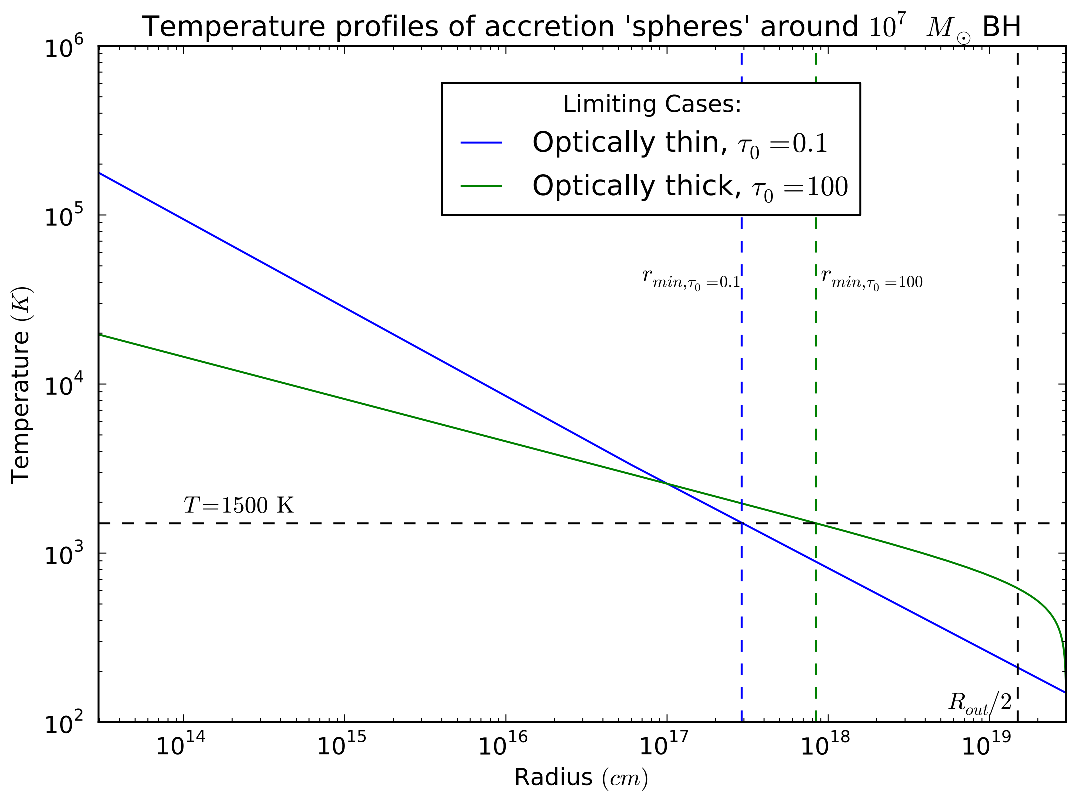
$$T_{equil}(r) = \left[\frac{3}{16\pi}\frac{\tau_0}{R_{out}}\frac{L_{BH}}{\sigma_{SB}}\left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}$$

But we also know that $L_{BH} = 4\pi {R_{in}}^2\sigma_{SB}T_s^4$, so we can rewrite this expression as

$$T_{equil}(r) = \left[\frac{3}{4(\cancel{4\pi})}\frac{\tau_0}{R_{out}}\frac{\cancel{4\pi}{R_{in}}^2\cancel{\sigma_{SB}}T_s^4}{\cancel{\sigma_{SB}}}\left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}$$

$$\boxed{T_{equil}(r) = T_s\left[\frac{3}{4}\frac{\tau_0}{R_{out}}{R_{in}}^2\left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}}$$

(f) Plotting the temperature profiles for the optically thin and the optically thick cases give us the profiles below, using $\tau_0 = 0.1$ for the optically thin case and $\tau + 0 = 100$ for the optically thick case.



Because dust sublimates for $T \gtrsim 1500\,\mathrm{K}$, we find that this happens at about

$$\boxed{r_{min,\tau_0=0.1} \approx 2.9 \times 10^{17}\,\mathrm{cm} = 0.10\,\mathrm{pc}}$$

$$\boxed{r_{min,\tau_0=100} \approx 8.4 \times 10^{17}\,\mathrm{cm} = 0.28\,\mathrm{pc}}$$

(g) Thinking about our dusty gas in the middle of the envelope at $r = R_{out}/2$, we compare the ratio for the temperatures for the optically thick and the optically thin cases.

$$\frac{T_{opt\ thick}(R_{out}/2)}{T_{opt\ thin}(R_{out}/2)} = \frac{T_s \left[\frac{3}{4}\frac{\tau_0}{R_{out}} {R_{in}}^2 \left(\frac{2}{R_{out}} - \frac{1}{R_{out}}\right)\right]^{1/4}}{T_s \left[\frac{1}{2}\left(1 - \left[1 - \frac{R_{in}^2}{(R_{out}/2)^2}\right]^{1/2}\right)\right]^{1/4}}$$

$$= \left[\frac{\frac{3}{4}\tau_0 \frac{{R_{in}}^2}{{R_{out}}^2}}{\frac{1}{2}\left(1 - \left(1 - \frac{4{R_{in}}^2}{{R_{out}}^2}\right)^{1/2}\right)}\right]^{1/4}$$